Linear hydrodynamics of protoplanetary disks: a crash course

Min-Kai Lin

@linminkai

July 2018
Outline

- Fluid instabilities in protoplanetary disks
- Linear stability analysis
- Aim: concepts and methods through examples
Planets form in accretion disks around young stars
Birthsites of planets: protoplanetary disks

Yen et al., 2016 (ASIAA)
Birthsites of planets: protoplanetary disks

Tang et al. 2017 (ASIAA)
Protoplanetary disks are dusty

- Protoplanetary discs are \(\sim 99\%\) gas, \(\sim 1\%\) dust
- Planets form from the solids (at least in core accretion)

(Testi et al., 2014)
Fluid dynamical instabilities provides a means to

- Develop turbulence for angular momentum transport and mass accretion
- Develop large-scale, potentially observable sub-structures (rings, gaps, asymmetries)
- Transport/collection of dust particles for planetesimal formation
Instabilities and PPD evolution

Examples in PPDs

- Gravitational instability
- Magneto-rotational instability (see M. Pessah’s talk)
- Vertical shear instability (see later)
- Rossby wave instability
- Convective overstability/baroclinic instability
- Elliptic instability
- Viscous overstability
- Zombie vortex instability
- Streaming instability
Rossby wave instability and vortex formation

- Disk-planet interaction $\rightarrow$ gap opening $\rightarrow$ unstable $\rightarrow$ vortices
Rossby wave instability and vortex formation

- Origin of dust-traps (OpH IRS 48, van der Marel, 2013)?
Vertical shear instability and dust settling

Initially well-mixed dust

Particles settle to mid-plane if the gas is laminar

vertical shear instability $\rightarrow$ hydro. turbulence $\rightarrow$ particles stirred up

$\frac{\rho_{\text{dust}}}{\rho_{\text{gas}}}$ shown
Vertical shear instability and dust settling

(HL Tau, Pinte et al. 2015)

- Observed thin dust layers consistent with weak turbulence
Streaming instability and planetesimal formation

- **Streaming instability** → dust clumping mediated by mutual dust-gas drag
Streaming instability and planetesimal formation

Streaming instability → dust clumping mediated by mutual dust-gas drag
Streaming instability and planetesimal formation

Early phase of the instability can be described by a linear analysis.

St=0.1, $\rho_d=3\rho_g$, $K_{x,z}=30$

Simulation
Linear theory

linear analysis
Linear analysis

- Full equations describing disk evolution is non-linear
  → need numerical simulations
- Linear analysis: evolution of small perturbations away from a known reference state, e.g.
  \[ \Sigma \rightarrow \Sigma_{\text{ref}} + \delta \Sigma(x, t) \]
  \[ \delta \Sigma: \text{perturbations (Eulerian today)} \]
  Assume \[|\delta \Sigma / \Sigma_{\text{ref}}| \ll 1\], ignore terms of \[O(|\delta \Sigma|^2)\]
  Useful for: gaining physical insight, code-testing
Generic procedure

1. Identify physics and geometry, write down the full governing equations
Generic procedure

1. Identify physics and geometry, write down the full governing equations

2. Important: solve governing equations for a well-defined basic state: often interested in steady equilibria with symmetries
Generic procedure

1. Identify physics and geometry, write down the full governing equations

2. Important: solve governing equations for a well-defined basic state: often interested in steady equilibria with symmetries

3. Insert $\Sigma \rightarrow \Sigma_{\text{ref}} + \delta \Sigma$ etc. into governing equations and linearize
Generic procedure

1. Identify physics and geometry, write down the full governing equations

2. **Important**: solve governing equations for a well-defined basic state: often interested in steady equilibria with symmetries

3. Insert $\Sigma \rightarrow \Sigma_{\text{ref}} + \delta \Sigma$ etc. into governing equations and linearize

4. Take Fourier dependence for co-ordinates in which the basic state is symmetric. For example, if $\partial_t \equiv 0, \partial_\phi \equiv 0$ for the basic state, then take

$$\delta \Sigma(r, \phi, z, t) = \text{Re} [\Sigma'(x, z) \exp(\text{i} m \phi - \text{i} \omega t)]$$

- Complex amplitude
- Can also take Fourier dependence when considering perturbations that vary much more rapidly than background, $|\partial_r \ln \Sigma_{\text{ref}}| \ll |k_r|$
- Fourier decomposition with non-uniform backgrounds possible for special perturbations

M-K. Lin (ASIAA)  July 2018 11 / 25
Generic procedure

1. Identify physics and geometry, write down the full governing equations.

2. Important: solve governing equations for a well-defined basic state: often interested in steady equilibria with symmetries.

3. Insert $\Sigma \rightarrow \Sigma_{\text{ref}} + \delta \Sigma$ etc. into governing equations and linearize.

4. Take Fourier dependence for co-ordinates in which the basic state is symmetric. For example, if $\partial_t \equiv 0, \partial_\phi \equiv 0$ for the basic state, then take

$$\delta \Sigma(r, \phi, z, t) = \text{Re} [\Sigma'(x, z) \exp (i m \phi - i \omega t)]$$

- Complex amplitude

5. Simplify/combine perturbation equations and solve for $\{\omega, \delta \Sigma, \delta \mathbf{v}\}$. 
Basic equations

- **Physics**
  - Hydrodynamics
  - Magneto-hydrodynamics
  - Full energy equation with heating/cooling/radiation, or fixed equation of state (polytropic, isothermal)
  - Gravity: star, planet, disk
  - Viscosity
  - Dust

- **Geometry**
  - Global: consider the entire disk
  - Local: follow a small patch of the disk

- **Dimensions**
  - 1D: azimuthally and vertically integrated
  - 2D: axisymmetric
  - 2D: razor-thin disk
  - 3D: full treatment
Basic equations

- **Physics**
  - Hydrodynamics
  - Magneto-hydrodynamics
  - Full energy equation with heating/cooling/radiation, or fixed equation of state (polytropic, isothermal)
  - Gravity: star, planet, disk
  - Viscosity
  - Dust

- **Geometry**
  - Global: consider the entire disk
  - **Local**: follow a small patch of the disk

- **Dimensions**
  - **1D**: azimuthally and vertically integrated
  - **2D**: axisymmetric
  - **2D**: razor-thin disk
  - **3D**: full treatment
Isothermal, self-gravitating shearing sheet

- Following a small patch of the disk as it rotates around the star (Goldreich & Lynden-Bell, 1965)
- Razor-thin disk model: material and velocity confined to $z = 0$ (but disk potential is still 3D)

**Governing equations**

\[
\begin{align*}
\frac{\partial \Sigma}{\partial t} + \nabla \cdot (\Sigma \mathbf{v}) &= 0, \\
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + 2\Omega \hat{\mathbf{z}} \times \mathbf{v} &= -\nabla \Phi_{\text{rot}} - \nabla \Phi_{d,z=0} - \frac{1}{\Sigma} \nabla P, \\
\nabla^2 \Phi_d &= 4\pi G \Sigma \delta(z),
\end{align*}
\]

with

\[
\begin{align*}
P &= c_s^2 \Sigma, \quad \Phi_{\text{rot}} = -\Omega S x^2.
\end{align*}
\]

$S =$ shear rate, $S = 3\Omega/2$ for a Keplerian disk
Isothermal, self-gravitating shearing sheet

- Following a small patch of the disk as it rotates around the star (Goldreich & Lynden-Bell, 1965)
- Razor-thin disk model: material and velocity confined to $z = 0$ (but disk potential is still 3D)

**Basic state**

\[ \partial_t = 0, \quad \partial_y = 0, \quad \Sigma_{\text{ref}} = \text{const.}, \quad P_{\text{ref}} = \text{const.}, \quad v_{\text{ref}} = -Sx\hat{y}. \]

- Generally can’t perform standard Fourier analysis in $x$ (need a modified Fourier analysis)
- Exceptions:
  - Non-shearing disk with $S = 0$ (i.e. solid body rotation)
  - Waves independent of $y$
Linearizing the shearing sheet equations

\[ \Sigma \rightarrow \Sigma + \delta \Sigma \]

...etc. Example

\[ \frac{\partial}{\partial t} (\Sigma + \delta \Sigma) + \nabla \cdot [(\Sigma + \delta \Sigma)(v + \delta v)] = 0. \]

Ignoring \( \delta^2 \) quantities and using \( \partial_t \Sigma + \nabla \cdot (\Sigma v) = 0 \) for basic state:

\[ \frac{\partial \delta \Sigma}{\partial t} + \Sigma \nabla \cdot \delta v + v \cdot \nabla \delta \Sigma = 0. \]
Linearizing the shearing sheet equations

Let
\[ \delta \Sigma(x, y, t) = \Sigma'(x) e^{i(k_y y - \omega t)}, \]
get

\[ i [k_y v_y(x) - \omega] \Sigma'(x) + \sum \left[ \frac{dv'_x(x)}{dx} + ik_y v'_y(x) \right] = 0. \]

- ODE with \( x \)-dependent coefficients \( \rightarrow \) can’t Fourier analyze (think \( Y'' + \sigma^2 Y = 0 \)).
- But \( x \)-dependence vanishes for axisymmetric waves and/or non-shearing disk \( (k_y v_y = 0) \)
  - Set \( \Sigma'(x) = \tilde{\Sigma} e^{i k_x x} \ldots \text{etc.} \)
Axisymmetric waves in the shearing sheet

Axisymmetric linear equations

\[ i k_x \Sigma \tilde{\nu}_x = i \omega \tilde{\Sigma}, \]

\[ i k_x \left( \tilde{\Phi}_{d,z=0} + c_s^2 \frac{\tilde{\Sigma}}{\Sigma} \right) - 2\Omega \tilde{\nu}_y = i \omega \tilde{\nu}_x, \]

\[ (2\Omega - S) \tilde{\nu}_x = i \omega \tilde{\nu}_y, \]

Perturbed Poisson equation for a thin disk (disk potential is still 3D)

\[ \frac{\partial^2 \tilde{\Phi}_d}{\partial z^2} - k_x^2 \tilde{\Phi}_d = 4\pi G \tilde{\Sigma} \delta(z) \]

For \( z \neq 0 \), solve

\[ \frac{\partial^2 \tilde{\Phi}_d}{\partial z^2} - k_x^2 \tilde{\Phi}_d = 0 \]

\[ \rightarrow \tilde{\Phi}_d = A \exp(-|k_x z|) \]
Axisymmetric waves in the shearing sheet

### Axisymmetric linear equations

\[ i k_x \Sigma \tilde{v}_x = i \omega \tilde{\Sigma}, \]

\[ i k_x \left( \tilde{\Phi}_{d,z=0} + c_s^2 \frac{\tilde{\Sigma}}{\Sigma} \right) - 2 \Omega \tilde{v}_y = i \omega \tilde{v}_x, \]

\[ (2 \Omega - S) \tilde{v}_x = i \omega \tilde{v}_y, \]

Apply jump condition at \( z = 0 \) to get

\[ \tilde{\Phi}_{d,z=0} = - \frac{2 \pi G \tilde{\Sigma}}{|k_x|}. \]
Axisymmetric waves in the shearing sheet

Eigenvlaue problem

\[
\begin{bmatrix}
0 & \, i k_x & \, 0 \\
\, i \left( c_s^2 \Sigma - \frac{2\pi G}{|k_x|} \right) & \, 0 & \, -2\Omega \\
0 & \, (2\Omega - S) & \, 0
\end{bmatrix}
\begin{bmatrix}
\tilde{\Sigma} \\
\tilde{v}_x \\
\tilde{v}_y
\end{bmatrix}
= i\omega
\begin{bmatrix}
\tilde{\Sigma} \\
\tilde{v}_x \\
\tilde{v}_y
\end{bmatrix}.
\]

- \([\tilde{\Sigma} \ \tilde{v}_x \ \tilde{v}_y]^T\) is the eigenvector
- \(i\omega\) the eigenvalue
- If \(\sigma \equiv \text{Im}(\omega) > 0\), mode grows like \(e^{\sigma t}\)
- Oscillation frequency \(\text{Re}(\omega)\)
Dispersion relation and Toomre instability

- Need $\det(M - i\omega I) = 0$ for non-trivial solution, get $\omega = \omega(k_x)$

\[
\omega^2 = \underbrace{\kappa^2}_{\text{rotation}} + \underbrace{c_s^2 k_x^2}_{\text{pressure}} - 2\pi G \Sigma |k_x|
\]

- $\kappa^2 = 2\Omega(2\Omega - S)$

- Examine limits
  - No pressure, no self-gravity: epicyclic oscillations with
    \[\omega = \pm \kappa\]
    (need $\kappa^2 > 0$ for stability)
  - Non-rotating, no self-gravity: sound waves with
    \[\omega = \pm c_s k_x\]
  - No self-gravity: sound waves modified by rotation
    \[\omega = \pm \sqrt{\kappa^2 + c_s^2 k_x^2}\]
Dispersion relation and Toomre instability

- Need \( \det(M - i\omega I) = 0 \) for non-trivial solution, get \( \omega = \omega(k_x) \)

\[
\omega^2 = \kappa^2 \text{rotation} + c_s^2 k_x^2 \text{pressure} - 2\pi G\Sigma |k_x| \text{self-gravity}
\]

- \( \kappa^2 = 2\Omega(2\Omega - S) \)
- Instability requires \( \omega^2(k_x) < 0 \). Consider \( \omega(k_x) = 0 \) to find

\[
Q \equiv \frac{\kappa c_s}{\pi G\Sigma} < 1 \quad \text{for instability.}
\]

- Critical wavenumbers:

\[
|k_x| = \frac{\pi G\Sigma}{c_s^2} \left[ 1 \pm \sqrt{1 - Q^2} \right]
\]

- Small \( k_x \): large-scales stabilized by rotation
- Large \( k_x \): small-scales stabilized by pressure
Write down the basic equations (Keplerian disk)

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\frac{1}{\rho} \nabla P - 2\Omega \mathbf{\hat{z}} \times \mathbf{v} + 3\Omega^2 \mathbf{x} \mathbf{\hat{x}} + \Omega^2 z \mathbf{\hat{z}} \\
\end{align*}
\]

approx. vertical stellar gravity

\[P = c_s^2 \rho.\]
Axisymmetric waves in the 3D shearing box

2. Work out equilibrium

\[ \rho(z) = \rho(0) \exp \left( -\frac{z^2}{2H^2} \right), \quad H \equiv \frac{c_s}{\Omega} \]

\[ \mathbf{v} = -\frac{3}{2} \Omega \hat{y} \]

- Stratified equilibrium depends on \( z \) → can’t Fourier analyze in \( z \)
Axisymmetric waves in the 3D shearing box

Linearize, with $\delta \rho = \tilde{\rho}(z) \exp(ik_x x - i\omega t)\ldots$ etc.

$$-i\omega \frac{W}{c_s^2} = -ik_x \tilde{\nu}_x - \frac{d \ln \rho}{dz} \tilde{\nu}_z - \frac{d\tilde{\nu}_z}{dz},$$

$$-i\omega \tilde{\nu}_x = -ik_x W + 2\Omega \tilde{\nu}_y,$$

$$-i\omega \tilde{\nu}_y = -\frac{\Omega}{2} \tilde{\nu}_x,$$

$$-i\omega \tilde{\nu}_z = -\frac{dW}{dz},$$

where $W(z) \equiv c_s^2 \tilde{\rho}(z)/\rho(z)$. 
Simplify/combine linearized equations

\[
\frac{d^2 W}{dz^2} + \frac{d \ln \rho}{dz} \frac{dW}{dz} + \omega^2 \left( \frac{1}{c_s^2} + \frac{k_x^2}{\Omega^2 - \omega^2} \right) W = 0.
\]
Simplify/combine linearized equations

\[
\frac{d^2 W}{dz^2} + \frac{d \ln \rho}{dz} \frac{dW}{dz} + \omega^2 \left( \frac{1}{c_s^2} + \frac{k_x^2}{\Omega^2 - \omega^2} \right) W = 0.
\]

- Try to analyze equation without solving it. Multiply by \( \rho W^* \) and integrate from \( z = z_1 \) to \( z = z_2 \). Integrate by parts and assume boundary terms vanish (no pert. there)

\[
\int_{z_1}^{z_2} \rho \left| \frac{dW}{dz} \right|^2 dz = \omega^2 \left( \frac{1}{c_s^2} + \frac{k_x^2}{\Omega^2 - \omega^2} \right) \int_{z_1}^{z_2} \rho |W|^2 dz.
\]

- Thus \( \omega^2 \) is real and \( > 0 \) (exercise)

system is stable to axisymmetric perturbations
Explicit solution

Inserting the equilibrium density profile, the governing equation becomes

\[
\frac{d^2 W}{dZ^2} - Z \frac{dW}{dZ} + \nu^2 \left(1 + \frac{K_x^2}{1 - \nu^2}\right) W = 0.
\]

\[Z = z/H, \ \nu = \omega/\Omega, \ K_x = k_x H\]
Explicit solution

Inserting the equilibrium density profile, the governing equation becomes

\[
\frac{d^2 W}{dZ^2} - Z \frac{dW}{dZ} + \nu^2 \left(1 + \frac{K_x^2}{1 - \nu^2}\right) W = 0.
\]

\[Z = \frac{z}{H}, \ \nu = \frac{\omega}{\Omega}, \ K_x = k_x H\]

- Hermite differential equation, but if we didn’t know this:
Explicit solution

Inserting the equilibrium density profile, the governing equation becomes

\[
\frac{d^2 W}{dZ^2} - Z \frac{dW}{dZ} + \nu^2 \left( 1 + \frac{K_x^2}{1 - \nu^2} \right) W = 0.
\]

\[Z = \frac{z}{H}, \ \nu = \frac{\omega}{\Omega}, \ K_x = k_x H\]

- Series solution (check for singularities, may need Frobenius method)

\[W(Z) = \sum_{n=1}^{\infty} W_n Z^n\]

- Recurrence relation

\[(n + 2)(n + 1) W_{n+2} + \left[ \nu^2 \left( 1 + \frac{K_x^2}{1 - \nu^2} \right) - n \right] W_n = 0.\]

- Boundary condition \(|\delta \rho| \propto e^{-Z^2/2}|W| \rightarrow 0\) as \(|z| \rightarrow \infty\)

\[\Rightarrow W\] is a finite polynomial, say of order \(N\) \((W \propto He_N)\)
Explicit solution

Inserting the equilibrium density profile, the governing equation becomes

\[
\frac{d^2 W}{dZ^2} - Z \frac{dW}{dZ} + \nu^2 \left( 1 + \frac{K_x^2}{1 - \nu^2} \right) W = 0.
\]

\[Z = \frac{z}{H}, \quad \nu = \frac{\omega}{\Omega}, \quad K_x = k_x H\]

- Dispersion relation

\[\nu^2 \left( 1 + \frac{K_x^2}{1 - \nu^2} \right) = N.\]

- So low frequency waves with small wavelength (|\nu| \ll 1, K_x \gg 1) obey

\[\nu^2 K_x^2 = N \text{ (inertial waves)}\]

- Inertial waves can be destabilized by vertical shear
Real life example: vertical shear instability

PPDs can exhibit vertical shear, $\partial_z \Omega \neq 0$, because $\nabla P \times \nabla \rho \neq 0$ (baroclinic).

Vertically isothermal thin-disk with $T \propto R^q$,

$$R \frac{\partial \Omega}{\partial z} \simeq \left( \frac{qz}{2H} \right) \times \frac{H}{R} \Omega_{\text{Kep}}$$

$h \equiv H/R \sim 0.05$ in PPDs, so vertical shear is weak.
Real life example: vertical shear instability

\[ \partial_z \Omega \neq 0 \Rightarrow \text{free energy} \to \text{instability?} \]

- Change in kinetic energy:

\[ \Delta E \sim l_R^2 \left( \Omega^2 + \frac{l_z}{l_R} \cdot R \frac{\partial \Omega^2}{\partial z} \right) \]

- Vertical shear is weak, but

\[ \Delta E < 0 \quad \text{if} \quad |l_z| \gg |l_r| \]

- Energy released for vertically elongated disturbances \Rightarrow \text{instability}
Real life example: vertical shear instability

- PLUTO simulation of the VSI, characteristic large-scale vertical velocities
Vertical shear instability: the analysis

System of interest: 3D, non-self-grav. disk with fixed temp. profile $c_s^2(R) \propto R^q$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P - \nabla \Phi_*, $$

$$P = c_s^2(R) \rho.$$ (global description for now)

Equilibria: axisymmetric, stratified disk

$$\partial_t = \partial_\phi = 0, \quad \rho = \rho(R, z), \quad \mathbf{v} = R\Omega(R, z)\hat{\phi}$$

Solve

$$\frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{\partial \Phi_*}{\partial z} = 0,$$

$$\frac{1}{\rho} \frac{\partial P}{\partial R} + \frac{\partial \Phi_*}{\partial R} = R\Omega^2.$$
Vertical shear instability: the analysis

System of interest: 3D, non-self-grav. disk with fixed temp. profile $c_s^2(R) \propto R^q$

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,
\]
\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P - \nabla \Phi_*,
\]
\[
P = c_s^2(R) \rho.
\]

(global description for now)

Equilibria: vertical shear

\[
R \frac{\partial \Omega^2}{\partial z} = \frac{qz}{R} \frac{\Omega_{\text{Kep}}^2}{(1 + z^2/R^2)^{3/2}}.
\]
Vertical shear instability: the analysis

Axisymmetric linearized equations

For $\delta \rho(R, z) = \tilde{\rho}(z) \exp(ik_x R - i\omega t)$ assuming large $|k_x R| \ll 1$ (motivated by sims.)

\[
i\omega W = ik_x \tilde{v}_x + \frac{d\tilde{v}_z}{dz} + \frac{\partial \ln \rho}{\partial z} \tilde{v}_z,
\]
\[
i\omega \tilde{v}_x = ik_x W - 2\Omega \tilde{v}_y,
\]
\[
i\omega \tilde{v}_y = R \frac{\partial \Omega}{\partial z} \tilde{v}_z + \frac{\kappa^2}{2\Omega} \tilde{v}_x,
\]
\[
i\omega \tilde{v}_z = \frac{dW}{dz}.
\]

- Solve for $u(z; \omega) = [W, \tilde{v}_x, \tilde{v}_y, \tilde{v}_z]^T$ subject to appropriate vertical boundary conditions (top and bottom)
- Eigenvalue problem because solution only exists for certain $\omega$ (think SHM).
Vertical shear instability: the analysis

\[
\frac{d^2 W}{dZ^2} - \left(1 + \frac{iqhK_x}{1 - \nu^2}\right) Z \frac{dW}{dZ} + \nu^2 \left(1 + \frac{K_x^2}{1 - \nu^2}\right) W = 0,
\]

- Can solve as before to get

\[
\nu^2 = N \left(\frac{1 + iqhK_x}{N + 1 + K_x^2}\right),
\]

for \(|\nu| \ll 1.

- \(\nu^2\) is generally complex \(\rightarrow\) there is an unstable solution with \(\text{Im}(\nu) > 0\).

- Integral relations \(\rightarrow\) growth rates \(\sigma < |R\partial_z \Omega|\)
Numerical solutions

- Not always possible to solve linearized equations analytically
- Example: analytic solution assumes a boundary condition at infinity, real disks may have finite thickness
- Numerical tools
  - Discretizing the equations: finite-difference, pseudo-spectral
  - Eigenvalue problem: ‘shooting’ method, matrix methods
Numerical solutions

Let

\[ W(Z) = \sum_{k=1}^{N_z} w_k \psi_k(Z) \]

where \( \psi_k \) are appropriate basis functions, e.g. Chebyshev polynomials

- Take vertical derivatives exactly.
- Demand linearized equations to be satisfied at a (special) set of \( z_j \).
- \( W(Z_j) \equiv W_j = \sum_k w_k \psi_k(Z_j) \) and \( W_j' = \sum_k w_k \psi'(Z_j) \) etc.,
- Linear equations becomes the generalized eigenvalue problem

\[ Lu = i\omega Tu \]

- Modify matrix elements to account for boundary conditions
- Solve with standard matrix packages, e.g. LAPACK
Numerical solutions

Example: the eigenvalue problem

$$\frac{d^2 y}{dx^2} = -k^2 y$$

is discretized to

$$\sum_{j=1}^{N} L_{ij} \psi''_{ij} y_j = -k^2 \sum_{j=1}^{N} T_{ij} \psi_{ij} y_j,$$

where $\psi_{ij} \equiv \psi_j(x_i)$, for interior points $j = 2$ to $j = N - 1$.

- Suppose the boundary conditions are $y(x_1) = y(x_N) = 0$. Can set $L_{1j} = \psi_{1j}$ and $L_{Nj} = \psi_{Nj}$
Example VSI solutions

- Roughly horizontal ‘body modes’ and nearly vertical ‘surface modes’
- Here $\sigma$ is growth rate, $\omega$ is real frequency
- Eigenvalues ‘off’ from analytical method related to finite domain size
Example VSI solutions

Eigenvector $v_z$ (fundamental mode)

$$\omega = -2.04h\Omega_K, \sigma=0.45h\Omega_K, \beta=0.001, k_xH=10.0$$

- Large-scale vertical motions
Summary

- Fluid instabilities play a major role in PPD gas (and dust) dynamics
- Linear analysis reveals the properties of waves and instabilities
- Important to have a well-defined basic state to perturb
- If possible, analyze linear equations first to infer properties of solutions (can be used to check numerical results)
- If possible, solve equations analytically to obtain explicit dispersion relations
- If needed, solve equations numerically
Potential projects

- Stability of stratified, dusty disks with self-gravity, particle diffusion, viscosity
- Floquet stability analysis of exact vortex solutions in magnetized disks
- Stability of dusty vortices
- Global stability analysis of dusty disks
- Hydrodynamic instabilities with self-gravity
- Dusty, magnetized disks
Further reading

- Chandrasekhar, Hydrodynamic and Hydromagnetic Stability
- Pringle & King, Astrophysical Flows
- Clarke & Carswell, Principles of Astrophysical Fluid Dynamics
- Ogilvie, Lecture Notes on Astrophysical Fluid Dynamics
- Klahr, Pfeil & Schreiber, 2018 (review on hydrodynamic instabilities in PPDs)