Type III migration in a low viscosity disc

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Outline

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Type III migration in an inviscid disc:
  - Formation of vortensity rings
  - Linear stability
  - Non-linear outcome and role in type III
Summary
Introduction

- 405 exo-planets discovered (27 November 2009).
- First ‘hot Jupiter’ around 51 Pegasi, orbital period 4 days (Mayor & Queloz 1995). Fomalhaut b with semi-major axis 115AU.
- Formation difficult in situ, so invoke migration: interaction of planet with gaseous disc (Goldreich & Tremaine 1979; Lin & Papaloizou 1986).
Consider a migrating Saturn-mass planets with partial gap, so there is flow of material across co-orbital radius.

- Fluid element orbital radius changes from $a - x_s \rightarrow a + x_s \Rightarrow$ torque on planet:

\[
\Gamma_3 = 2\pi a^2 \dot{a} \Sigma_e \Omega x_s.
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  $$\Gamma_3 = 2\pi a^2 \dot{a}\Sigma_e \Omega x_s.$$  

- Migration rate for $M_p + M_r + M_h$:
  $$\dot{a} = \frac{2\Gamma_L}{\Omega a (M_p + M_r - \delta m)}$$  

where $\Gamma_L$ is the total Lindblad torque and

$$\delta m = 4\pi \Sigma_e ax_s - M_h = 4\pi ax_s (\Sigma_e - \Sigma_g)$$

is the density-defined co-orbital mass deficit.
Key ideas in type III

- Co-orbital mass deficit: larger $\delta m \Rightarrow$ faster migration.

- Horse-shoe width: $x_s$, separating co-orbital and circulating region. Take $x_s = 2.5r_h$ for result analysis ($r_h \equiv (M_p/3M_*)^{1/3}a$). Can show $x_s \lesssim 2.3r_h$ in particle dynamics limit.

- Vortensity: $\eta \equiv \omega/\Sigma$, important for stability properties and $\eta^{-1}$ also used to define $\delta m$ (Masset & Papaloizou 2003).

- Modelling assumptions: steady, slow migration ($\tau_{\text{lib}}/\tau_{\text{mig}} \ll 1$), horse-shoe material moves with planet.
Standard numerical setup for disc-planet interaction. 2D disc in polar co-ordinates centered on primary but non-rotating. Units $G = M_\ast = 1$.

- Hydrodynamic equations with local isothermal equation of state:

\[
\frac{\partial \Sigma}{\partial t} + \nabla \cdot (\Sigma \mathbf{v}) = 0,
\]
\[
\frac{\partial v_r}{\partial t} + \mathbf{v} \cdot \nabla v_r - \frac{v_\phi^2}{r} = -\frac{1}{\Sigma} \frac{\partial P}{\partial r} - \frac{\partial \Phi}{\partial r} + \frac{f_r}{\Sigma},
\]
\[
\frac{\partial v_\phi}{\partial t} + \mathbf{v} \cdot \nabla v_\phi + \frac{v_\phi v_r}{r} = -\frac{1}{\Sigma r} \frac{\partial P}{\partial \phi} - \frac{1}{r} \frac{\partial \Phi}{\partial \phi} + \frac{f_\phi}{\Sigma},
\]
\[
P = c_s^2(r) \Sigma.
\]

Viscous forces $f \propto \nu = \nu_0 \times 10^{-5}$, temperature $c_s^2 = h^2/r$, $h = H/r$. $
\Phi$ is total potential including primary, planet (softening $\epsilon = 0.6H$), indirect terms but no self-gravity.

- Method: FARGO code (Masset 2000), finite difference for hydrodynamics, RK5 for planet motion.
Type III migration as a function of viscosity

Discs: uniform density $\Sigma = 7 \times 10^{-4}$, aspect ratio $h = 0.05$ and different uniform kinematic viscosities. Note $\nu = 10^{-5}$ equivalent to $\alpha_{SS} = 4 \times 10^{-3}$ at $r = 1$.

Planet: Saturn mass $M_p = 2.8 \times 10^{-4}$ initially at $r = 2$.

What's going on at low viscosities?
Inviscid case: evolution of $\Sigma/\omega$:
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Vortensity generated as fluid elements U-turn during its horse-shoe orbit.
Predicting the vortensity jump

We need:

- **Vortensity jump** across isothermal shock:

\[
\begin{bmatrix} \omega \\ \Sigma \end{bmatrix} = \left( \frac{M^2 - 1}{\Sigma M^4} \right) \frac{\partial v_\perp}{\partial S} - \left( \frac{M^2 - 1}{\Sigma M^2 v_\perp} \right) \frac{\partial c_s^2}{\partial S}.
\]

RHS is pre-shock. \( M = v_\perp / c_s \), \( S \) is distance along shock (increasing radius). Additional baroclinic term compared to Li et al. (2005) but has negligible effect (\( c_s^2 \propto 1/r \)).

- **Flow field**: shearing-box geometry, velocity field from zero-pressure momentum equations, density field from vortensity conservation following a particle.

- **Shock location**: generalised Papaloizou et al. (2004)

\[
\frac{dy_s}{dx} = \frac{\hat{v}_y^2 - 1}{\hat{v}_x \hat{v}_y - \sqrt{\hat{v}_x^2 + \hat{v}_y^2 - 1}}.
\]

\( \hat{v} \equiv v / c_s \).
Theoretical jumps

- Vortensity generation near shock tip (horse-shoe orbits), vortensity destruction further away (circulating region). Variation in flow properties on scales of $r_h \approx H$.
- Variation in disc profiles on scale-heights enables shear instability $\Rightarrow$ vortices in non-linear stage (Lovelace et al. 1999, Li et al. 2001).
Idea: linear stability analysis of inviscid disc but use simulation vortensity profile as basic state: axisymmetric, $v_r = 0$. Basic state checked explicitly.

Governing equation for isothermal perturbations $\propto \exp i(\sigma t + m\phi)$:

$$\frac{d}{dr} \left( \frac{\Sigma}{\kappa^2 - \bar{\sigma}^2} \frac{dW}{dr} \right) + \left\{ \frac{m}{\bar{\sigma}} \frac{d}{dr} \left[ \frac{\kappa^2}{r\eta(\kappa^2 - \bar{\sigma}^2)} \right] - \frac{r\Sigma}{h^2} - \frac{m^2\Sigma}{r^2(\kappa^2 - \bar{\sigma}^2)} \right\} W = 0$$

$W = \delta \Sigma / \Sigma$; $\kappa^2 = 2\Sigma \eta \Omega$; $\bar{\sigma} = \sigma + m\Omega(r)$.

Self-excited modes in inviscid disc with sharp vortensity profiles.
Example: \( m = 3, h = 0.05 \)

- Disturbance focused around vortensity minimum (gap edge), exponential decays either side joined by vortensity term at co-rotation \( r_0 \). More extreme minimum \( \Rightarrow \) more localised.
- Waves beyond the Lindblad resonances \( (\kappa^2 - \bar{\sigma}^2 = 0) \) but amplitude not large compared to co-rotation.
Link to type III migration

Recall co-orbital mass deficit

\[ \delta m = 4\pi a x_s (\Sigma_e - \Sigma_g) \]

- Instability can increase \( \delta m \) by increasing \( \Sigma_e \) but not \( \Sigma_g \) (co-rotational modes are localised) \( \Rightarrow \) favouring type III. When vortex flows across co-orbital region, \( \Sigma_g \) increases and migration may stall.
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- Can expect interaction when \( \delta m \) of order planet mass.
Growth rate independent of density scale. Higher density just means less time needed for vortex to grow sufficiently large for interaction.
\[ c_s^2 = T \propto h^2. \] Lower temperature \( \Rightarrow \) stronger shocks \( \Rightarrow \) profile more unstable \( \Rightarrow \) shorter time-scale to vortex-planet interaction.

Require disc profile to be sufficiently extreme and have enough mass to trigger vortex-planet interaction, but the extent of migration during one episode is the same.
Migration in low viscosity/inviscid discs is non-smooth due to shear instabilities associated with gap edge (vortensity minima). Effects appear at $\nu = O(10^{-6})$ or $\alpha_{SS} = O(10^{-4})$.

Provided an over-all picture of vortex-planet interaction: formation of unstable basic state via shocks, linear stability analysis and hydrodynamic simulations.

Instability favours type III migration by increasing the co-orbital mass deficit. Vortex-planet interaction when $\delta m/M_p \sim 4-5$. Associated disruption of co-orbital vortensity structure, unlike previous notion of type III migration where flow-through does not affect co-orbital region.