Linear vertical shear instability in protoplanetary disks

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Outline

1. Results
2. Isothermal linear theory
3. Linear theory with finite cooling
4. Numerical calculations
5. Application to the MMSN
Thermodynamic condition for the VSI

- Astrophysical disks generally have $\partial_z \Omega \neq 0$ — necessary for VSI, but also need rapid cooling.

(Nelson et al., 2013)

Can we quantify this requirement?
Lin-Youdin VSI condition

\[ t_{\text{cool}} \Omega_K < \frac{h|q|}{\gamma - 1} \equiv \beta_{\text{crit}} \]

(Vertically isothermal disk with \( T \propto r^q \), \( h \equiv H/r \), and \( t_{\text{cool}} = \beta/\Omega_K \).)

(Lin & Youdin, 2015)
Rapid cooling needed because of buoyancy
Vertical motion associated with VSI is opposed by buoyancy forces

\[ \frac{r\partial_z \Omega}{N_z} \]

destabilizing vert. shear

v.s.

stabilizing vert. buoyancy

Vertical shear is weak, \[ r\partial_z \ln \Omega \sim O(h) \ll 1 \], so need \( l_z/l_r \gg 1 \)

Vertical buoyancy is strong, \[ N_z/\Omega \sim O(1) \]
Linear theory: previous analyses and our contribution

- Vertically and radially local, with energy equation
  (Urpin & Brandenburg, 1998; Urpin, 2003; G. Mohandas)

- Vertically global, radially local, no buoyancy
  (Nelson et al., 2013; McNally & Pessah, 2014; Barker & Latter, 2015)

- Vertically and radially global, no buoyancy
  (Barker & Latter, 2015; Umurhan et al., 2015)

Lin & Youdin (2015)

- Vertically global, radially local, including energy equation (i.e. with buoyancy)
- Both constant cooling and realistic cooling functions
Isothermal limit (instantaneous cooling)

Linearized fluid equations →

\[ 0 = W'' + \left[ \ln \rho' - \frac{iK}{(1 - \nu^2)\Omega_K^2 h} \frac{d\Omega^2}{dz} \right] W' + \nu^2 \left( 1 + \frac{K^2}{1 - \nu^2} \right) W. \]

\[ K = k_x H, \quad \nu = \omega / \Omega_K. \]

- **Formal**\(^1\) limit on the growth rate of low-frequency modes

\[ \sigma < \max \left| r \frac{d\Omega}{dz} \right| \]

(Unbound if approximating vertical shear \(\propto z\))

- General frequency waves in a thin-disk **without a surface**

\[ \nu^4 - (L + 1 + K^2) \nu^2 + L (1 + ihqK) = 0, \quad L = 1, 2 \cdots \]

VSI is the low-frequency (inertial) branch.

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\(^1\)Via Cauchy-Schwarz inequality...etc.
Linear theory with finite cooling

- Parameterized cooling: $t_{\text{cool}} \Omega_K \equiv \beta = \text{const.}$

Single ODE reduced model (low-freq., thin-disk, no explicit $\partial_r P$)

$$0 = \delta v_z''(z) - zA \delta v_z'(z) + (B - Cz^2) \delta v_z(z).$$
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- Transformation $\rightarrow$ Hermite ODE (as before)
- As in Lubow & Pringle (1993) but $A, B, C$ now complex because $\partial_z \Omega \neq 0$
- Important: reduced model is only valid for $t_{\text{cool}} \Omega_K \lesssim O(1)$ (OK for VSI)
Linear theory with finite cooling

- Parameterized cooling: \( t_{\text{cool}} \Omega_k \equiv \beta = \text{const.} \)

Single ODE reduced model (low-freq., thin-disk, no explicit \( \partial_r P \))

\[
0 = \delta v_z''(z) - zA\delta v_z'(z) + (B - Cz^2) \delta v_z(z).
\]

- Finite K.E. density as \( |z| \to \infty \Rightarrow \) dispersion relation \( \omega = \omega(k_x; \beta, M) \)
- Mode number \( M = 0, 1, 2 \ldots \)
- Fundamental mode \( M = 0 \) has special importance

![Graph](image-url)
Critical cooling time

- Assume $\beta = \beta_c$ at marginal stability ($\sigma = 0$) and large $k_x$

Find

$$\frac{\partial \beta_c}{\partial M} < 0$$

(if the disk is sufficiently thin). Then $M = 0$ has the longest critical cooling time.

The fundamental mode is the most difficult to stabilize with increasing $t_{cool}$. 
Critical cooling time

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So condition for VSI is

$$t_{\text{cool}} \Omega_K < \beta_c(M = 0) = \frac{h|q|}{\gamma - 1}$$

- $h|q|$: vertical shear (destabilizing)
- $\gamma - 1$: vertical buoyancy (stabilizing)
Numerical calculations

- Solve linearized equation in the radially local approx.
- Relax all other assumptions in reduced model

Theory describes the lowest order modes inc. fundamental mode
- ‘Surface modes’ are entirely due to disk surface (imposed or physical)
Effect of increasing the cooling time

\[ \frac{\sigma}{(\hbar \Omega_K)} \]

\[ -\frac{\omega}{(\hbar \Omega_K)} \]

- \( k_x H = 10, \ z_{\text{max}} = 5H \)
- \( \beta = 0.01 \)
- \( \beta = 0.03 \)
- \( \beta = 0.05 \)
- \( \beta = 0.10 \)
- \( M = 0, \text{analytic} \)
- \( M = 1, \text{analytic} \)
- \( M = 2, \text{analytic} \)
- \( M = 3, \text{analytic} \)

M-K. Lin, A. Youdin (Arizona)
Testing the critical cooling timescale

\[ t_{\text{cool}} \Omega_K < \frac{h|q|}{\gamma - 1} \]

Max. growth rate in \( h\Omega_{\text{Kep}} \) vs. Dimensionless cooling time \( \beta \)

- \( k_x H = 100 \)
- \( k_x H = 50 \)
- \( k_x H = 30 \)
- \( k_x H = 10 \)
- \( k_x H = 5 \)
- \( k_x H = 1 \)
Testing the critical cooling timescale

\[ t_{\text{cool}} \Omega_K < \frac{h|q|}{\gamma - 1} \]
Application to protoplanetary disks

Estimate cooling times in the Minimum Mass Solar Nebula (Chiang & Youdin, 2010) based on dust opacity ($\propto T^2$):

$$t_{\text{cool}} \Omega_K \equiv \beta(z; r, K) = 3.9 \times 10^{-3} \frac{r_{\text{AU}}^{9/14}}{\kappa_d} \left[ 1 + \frac{1.9 \times 10^7 \kappa_d^2}{r_{\text{AU}}^{33/7} K^2} \exp\left( -\frac{z^2}{2H^2} \right) \right]$$

- $\kappa_d$: opacity scale relative to MMSN
- Optically thin/Newtonian cooling for very small scales, fast for large $\kappa_d$
- Radiative diffusion for longer scales, fast for small $\kappa_d$
- Vert. dependence through $\rho$
Application to protoplanetary disks

Estimate cooling times in the Minimum Mass Solar Nebula (Chiang & Youdin, 2010) based on dust opacity ($\propto T^2$):
$\beta$ versus $\beta_{\text{crit}}$

$$\beta = t_c \Omega_K$$

MMSN opacity, $z=0$

- $\beta_{\text{crit}}$
- $\beta, k_x H=1$
- $\beta, k_x H=10$
- $\beta, k_x H=100$

$r$/AU

$\beta$ versus $\beta_{\text{crit}}$
\[ \beta \text{ versus } \beta_{\text{crit}} \]

\[ \beta = t_c \Omega K \]

0.1×MMSN opacity, z=0
$\beta$ versus $\beta_{\text{crit}}$

MMSN opacity, $z=0$

$\beta = t_c \Omega_K$

- $\beta_{\text{crit}}$
- $\beta, k_x H=1$
- $\beta, k_x H=10$
- $\beta, k_x H=100$

$r/\text{AU}$
VSI in the solar nebula

With $\beta = \beta(z; r, K)$
Further applications and extensions

- Use $\beta_{\text{crit}}$ as a simple, first ‘go to’ criteria to assess stability against VSI.
- Enroll $\beta_{\text{crit}}$ in 1D accretion models, e.g. $\alpha_{\text{VSI}}(t_{\text{cool}}, \beta_{\text{crit}})$. (Cf. GI stress from Toomre parameter.)
Further applications and extensions

- Use $\beta_{\text{crit}}$ as a simple, first ‘go to’ criteria to assess stability against VSI
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- Radially-global problem (with O. Umurhan)
- Non-axisymmetric problem
- Other instabilities are supported in the current model, e.g. convective overstability (Klahr & Hubbard, 2014; Lyra, 2014)
Conclusions

Lin-Youdin criterion

\[ t_{\text{cool}} \Omega_K < \frac{h|q|}{\gamma - 1} \]

- Astrophysical disks generally have \( \partial_z \Omega \neq 0 \)
- Thin PPDs are unstable if buoyancy ineffective: 
  \( N_z = 0 \) and/or \( t_{\text{cool}} \Omega_K \ll 1 \)
- Fast cooling needed because vertical shear is weak but buoyancy is strong
- Thermodynamic requirement satisfied at 10s of AU in typical PPDs
References