Large-scale hydrodynamic instabilities in protoplanetary disks

Min-Kai Lin
mklin924@cita.utoronto.ca

Canadian Institute for Theoretical Astrophysics

April 23 2013
Observational motivation

(Brown et al., 2009)

(Mayama et al., 2012)
Theoretical motivations

- Angular momentum transport: by vortices and non-local transport by waves
- Dust concentration by vortices $\rightarrow$ planetesimal formation
- Modifying planet migration
- Instabilities may be naturally associated with disk structure
  e.g. planet gaps and ‘dead zones’ $\rightarrow$ localized radial gradients
Theoretical motivations

(Armitage, 2011)
Toy model: axisymmetric over-dense ring
Rossby wave instability

- Kelvin-Helmholtz instability in a rotating disk (Lovelace et al., 1999)
- Thin-disk version of the Papaloizou-Pringle instability (Papaloizou & Pringle, 1985)

(Meheut et al., 2013)
Non-linear examples

ATHENA code: 3D disk in a Cartesian box
Non-linear examples

ZEUS code: 3D self-gravitating adiabatic disk
Non-linear examples

PLUTO code

3D disk with viscosity jump in radius
3D self-gravitating disk-planet simulation

[Note: global simulations plotted in a box \((r \rightarrow x, \phi \rightarrow y)\)]
Application I: angular momentum transport

Lyra & Mac Low (2012): non-ideal MHD simulation with jump in resistivity to mimic the dead zone/active zone boundary → vortex formation in DZ

![Graphs showing Maxwell stress and Reynolds stress in active and dead zones.](image-url)
Application II: planetesimal formation
Meheut et al. (2012): add dust to RWI-unstable disk
Starting point: linear stability

Linear problem by Lovelace et al. (1999):

adiabatic non-self-gravitating 2D disk

Recent generalizations:
- Self-gravity 2D (Lin & Papaloizou, 2011a,b; Lovelace & Hohlfeld, 2013)
- Magnetic fields 2D (Yu & Li, 2009; Yu & Lai, 2013)
- Isothermal 3D (Meheut et al., 2012)

This talk:
- Polytropic 3D (Lin, 2012a, 2013a)
- Adiabatic 3D (Lin, 2013b)
Starting point: linear stability

After some manipulation, we have the basic equation for $\chi(= \delta p / \rho)$ as

\[
\left[ \frac{\partial}{\partial r} \left( a_{rr} \frac{\partial}{\partial r} + a_{rz} \frac{\partial}{\partial z} + b_r \right) + \frac{\partial}{\partial z} \left( a_{zz} \frac{\partial}{\partial z} + a_{rz} \frac{\partial}{\partial r} + b_z \right) + d_r \frac{\partial}{\partial r} + d_z \frac{\partial}{\partial z} + f \right] \chi = 0,
\]

with

\[
a_{rr} = \frac{\rho \sigma r}{D} \left( 1 + \frac{\mu g_r^2}{D H} \right), \quad a_{zz} = \frac{\rho r}{\sigma} \left( 1 + \frac{\mu g_z^2}{\sigma^2 H} \right), \quad a_{rz} = \frac{\mu \rho g_r g_z r}{D H \sigma},
\]

\[
b_r = \frac{\mu \rho g_r}{D H} \left( \sigma r - \frac{2m \Omega g_r}{D} \right) - \frac{2m \Omega \rho}{D}, \quad b_z = \frac{\mu \rho g_z r}{\sigma H} \left( 1 - \frac{2m \Omega g_r}{\sigma D r} \right),
\]

\[
d_r = \frac{m \kappa^2 \rho}{2 \Omega D} - \left( \sigma r - \frac{m \kappa^2 g_r}{2 \Omega D} \right) \frac{\mu \rho g_r}{D H}, \quad d_z = - \left( \sigma r - \frac{m \kappa^2 g_z}{2 \Omega D} \right) \frac{\mu \rho g_z}{\sigma^2 H},
\]

\[
f = - \frac{m^2 \sigma \rho}{D r} - \left( \sigma r - \frac{m \kappa^2 g_z}{2 \Omega D} \right) \left( 1 - \frac{2m \Omega g_r}{D \sigma r} \right) \frac{\mu \rho}{H} + \frac{(\mu + 1) \sigma \rho}{c^2},
\]

(Kojima et al., 1989)
Linear problem for 3D polytropic disks \( (\rho \propto \rho^{1+1/n}) \)

1. Steady, axisymmetric, vertically hydrostatic density bump at \( r = r_0 \)
2. Perturb fluid equations, e.g. \( \rho \rightarrow \rho + \delta \rho(r, z) \exp(i(m\phi + \sigma t)) \)
3. Combine linear equations to get equation for \( W = \delta p / \rho \):

\[
L(r, z; \sigma)W = 0.
\]

- \( W \rightarrow \) eigenfunction; \( \sigma \rightarrow \) eigenvalue
- Note: \( \sigma \) appears through \( \bar{\sigma} = \sigma + m\Omega(r) \)
- RWI: \( \text{Re}[\bar{\sigma}(r_0)] \simeq 0 \) and \( \frac{d\eta}{dr}\bigg|_{r_0} \simeq 0 \) \( (\eta = \kappa^2 / 2\Omega \Sigma \) is the vortensity)

Very complicated PDE even for numerical work!
Application of orthogonal polynomials

$L(r, z; \sigma)$ only depends on $z$ through $\rho(r, z)$. For thin polytropic disks:

$$\rho(r, z) = \rho_0(r) \left[ 1 - \frac{z^2}{H^2(r)} \right]^n.$$
Application of orthogonal polynomials

$L(r, z; \sigma)$ only depends on $z$ through $\rho(r, z)$. For thin polytropic disks:

$$\rho(r, z) = \rho_0(r) \left[ 1 - \frac{z^2}{H^2(r)} \right]^n.$$ 

Ansatz:

$$W(r, z) = \sum_{l=0}^{\infty} W_l(r) C_l^\lambda(z/H),$$

where $C_l^\lambda(x)$ are Gegenbauer polynomials (generalization of Legendre and Chebyshev polynomials). Then

$$L(r, z; \sigma) W = 0 \rightarrow A_l(W_l) + B_l(W_{l-2}) + C_l(W_{l+2}) = 0,$$

where differential operators $A_l$, $B_l$ and $C_l$ only depend on $r$ and $\sigma$$\rightarrow$ vertical dependence exactly removed, but this is a lot of work!
Example problem

\( n = 1.5 \) polytrope with a surface density bump

Recall \( \eta = \frac{1}{r \Sigma} \frac{d}{dr} (r^2 \Omega) \) is the potential vorticity (note: RWI for PV minima only)
Example solution

\[ W(r, z) = W_0(r) + W_2(r)C_2^\lambda(z/H) + \cdots \]

Growth rate \( \sim 0.1\Omega \), same as 2D \( l_{\text{max}} \equiv 0 \). Instability is 2D.
Example solution

\[ W(r, z) = W_0(r) + W_2(r) C^\lambda_2(z/H) + \cdots \]

Note

\[ \delta v_z = i \left( \frac{\partial W}{\partial z} \right) / \tilde{\sigma} \text{ but } |\tilde{\sigma}| \sim 0 \text{ at } r \sim r_0 \]
Horizontal flow

Anti-cyclonic motion associated with over-density
Motion is upwards at \((r_0, \phi_0, z)\).
Vertical motion

Upward motion seen in non-linear hydrodynamic simulations of Meheut et al. (2012):

Meheut et al. (2012) → mm dust lifted to disk surface
Back to linear problem: equation of state

Magnitude of vertical motion decreases with increasing $n$ (more compressible)

$\leftarrow n = 1.0$ polytrope

$\leftarrow$ vertically isothermal disk ($n = \infty$, special treatment with Hermite polynomials)
Extension to adiabatic 3D disks

- $p \propto \rho \Gamma$ in basic state only
- Energy equation $Ds/Dt = 0$, $s \equiv p/\rho^\gamma \propto \rho^{\Gamma-\gamma}$
- $\gamma \geq \Gamma \geq 1$, density bump $\rightarrow$ entropy dip

\[
V_1 W + \bar{V}_1 Q = 0 \\
V_2 W + \bar{V}_2 Q = 0
\]

- $W = \delta p/\rho \rightarrow$ pressure perturbation
- $Q = c_s^2 \delta \rho/\rho \rightarrow$ density perturbation
- $S \equiv W - Q \rightarrow$ entropy perturbation
What should we look for?

\[ \bar{S} \equiv Q - \frac{\gamma}{\Gamma} W = \left(1 - \frac{\gamma}{\Gamma}\right) W - S \]
What should we look for?

\[ \tilde{S} \equiv Q - \frac{\gamma}{\Gamma} W = \left(1 - \frac{\gamma}{\Gamma}\right) W - S \]
Expectation and reality

\( \bar{S} \leq 0 \) along dotted line

\( \bar{S} \) decreasing

\( w = (\Gamma / \gamma) Q \)

so \( \bar{S} = 0 \) here

\( W, Q, S > 0 \)

so \( \bar{S} < 0 \) here

\( m = 3, \ Q = (\gamma / \Gamma) W \)

\( \bar{S} \Rightarrow \delta v_z \)

\( \nabla \bar{S} \Rightarrow (\nabla \times \delta \mathbf{v})_\phi \)
PDE eigenvalue problem: numerical approach

Finite-difference in \( r \), pseudo-spectral in \( Z \equiv z/H \):

\[
W(r_i, z) \equiv W_i(Z) = \sum_{k=1}^{N_z} w_{ki} \psi_k(Z/Z_{\text{max}})
\]

\[
[V_1 - \bar{V}_1(\bar{V}_2^{-1} V_2)] W = 0 \rightarrow U(\sigma)w = 0
\]

- \( U \rightarrow \) matrix representation of PDE operator
- \( w \rightarrow \) vector to store the \( w_{ki} \)
- Vertical boundary condition: \( \Delta P = 0, \delta v_z = 0 \) or \( \delta v_\perp = 0 \) at \( Z = Z_{\text{max}} \)
- *Much easier* to derive and implement than previous method, and allows for different vertical b.c., but need an accurate initial guess for \( \sigma \)

See Lin (2013a) for method recipe.
Non-homentropic example

$\Gamma = 1.67, \gamma = 2.5$

$N$ is the buoyancy frequency
Non-homentropic example

\( \Gamma = 1.67, \gamma = 2.5, m = 3 \) along \( \phi = \phi_0 \).

Growth rate 0.1099\( \Omega_0 \) (cf. 0.1074\( \Omega_0 \) for \( \gamma = 1.67 \))
Entropy perturbation

$\Gamma = 1.67$, $\gamma = 2.5$, $m = 3$
Meridional vortical motion

\[ \Gamma = 1.67, \; \gamma = 2.5, \; m = 5 \text{ along } \phi = \phi_0 \]
Vertical motion

Fix $\Gamma = 1.67$, vary $\gamma$, plot $\delta v_z$ along $(r_0, \phi_0, z)$. 

![Graph showing vertical motion, with different lines representing different values of $\Gamma$ and $\gamma$.]
Vertical motion

Kato (2001):

\[
\delta v_z \sim -\frac{\nu}{N_z^2} \frac{\partial W}{\partial z} - \nu \rho \left( \frac{\partial p}{\partial z} \right)^{-1} - \frac{1}{\nu} \frac{\partial W}{\partial z}, \quad N_z^2 \neq 0
\]

at co-rotation radius, and \( \nu \) here is the growth rate. Compared to

\[
\delta v_z \sim -\frac{1}{\nu} \frac{\partial W}{\partial z}, \quad N_z^2 \equiv 0.
\]

Notice for \( N_z^2 \neq 0 \)

\[
\text{pressure buoyancy} \sim \frac{\Omega^2}{N_z^2} \frac{\partial \ln W}{\partial \ln z},
\]

i.e. buoyancy dominates at large \( z \) as \( N_z^2 \) increases with height.

Origin of \( \delta v_z \) is different between homentropic and non-homentropic flow
Comparison with hydrodynamic simulations

- Isothermal disk, adiabatic evolution ($\Gamma \equiv 1, \gamma = 1.4$)

ZEUS simulation

$[h_{iso \tan \theta}]^{-1} = 0.0$ (nonlinear)

Linear code

$z/H_{iso} = 0.0$ (linear)
Comparison with hydrodynamic simulations

- Isothermal disk, adiabatic evolution ($\Gamma \equiv 1$, $\gamma = 1.4$)

ZEUS simulation

Re($\sigma$) = $-0.99m\Omega_0$
Im($\sigma$) = $-0.194\Omega_0$

Linear code

Re($\sigma$) = $-0.9896m\Omega_0$
Im($\sigma$) = $-0.1937\Omega_0$
Vortex-formation in layered-accretion disks?

PLUTO disk-planet experiments

Imposed viscosity $\alpha \sim 10^{-4}$ everywhere

[Lin and Umurhan (in preparation)]
Vortex-formation in layered-accretion disks?

PLUTO disk-planet experiments

\[ \log \left( \frac{\nu}{r_0^2 \Omega_k(r_0)} \right) \]

\[ \alpha \sim 10^{-2} \]

\[ \alpha \sim 10^{-4} \]

[Lin and Umurhan (in preparation)]]
Vortex-formation in layered-accretion disks?

PLUTO disk-planet experiments

$\alpha \sim 10^{-4}$ in bulk of the disk, $\alpha \sim 10^{-2}$ in atmosphere

$[\text{Lin and Umurhan (in preparation)}]$
Self-gravity

- Vortices are over-dense blobs
- Vortensity \( \eta \) and Toomre \( Q_T \) are related: \( Q_T = \left( \frac{c_s}{\pi G} \right) \sqrt{2 \Omega \eta / \Sigma} \)
- **Stabilization** of low \( m \) vortex modes, see Lin & Papaloizou (2011a) for formal proof and linear calculations

The 2D linear problem with self-gravity:

\[
L(S) = \delta \Sigma, \quad S = c_s^2 \delta \Sigma / \Sigma + \delta \Phi.
\]

\[
\int r S^* L(S) dr = \int r S^* \delta \Sigma dr = \text{energy}.
\]

For modes associated with vortensity extrema:

\[
\int \frac{m |S|^2}{\bar{\sigma}} \frac{d}{dr} \left( \frac{1}{\eta} \right) dr \quad \sim \quad \int rc_s^2 \frac{\delta \Sigma^2}{\Sigma} dr + \int r \delta \Phi^* \delta \Sigma dr
\]

\( > 0 \) for \( \min(\eta) \) at \( r = r_c \) (RWI)

thermal energy \( > 0 \)

gravitational energy \( < 0 \)

Balance does not work for strong SG (RHS < 0, gravitational disturbance)
Self-gravity

- Vortices are over-dense blobs
- Vortensity $\eta$ and Toomre $Q_T$ are related: $Q_T = (c_s/\pi G)\sqrt{2\Omega \eta/\Sigma}$
- Stabilization of low $m$ vortex modes, see Lin & Papaloizou (2011a) for formal proof and linear calculations

Self-gravity in 3D [$\min(Q_T) = 8$]:

(Global 3D ZEUS simulations, Lin, 2012b). What about massive disks?
Gravitational edge instabilities

GI associated with gaps or edges even when Toomre stability criterion satisfied ($Q_T > 1$ everywhere)

- Lovelace & Hohlfeld (1978); Sellwood & Kahn (1991): galactic/stellar disks
- Meschiari & Laughlin (2008): gaps in gaseous protoplanetary disks
- Lin & Papaloizou (2011b): confirmation of GEI for planet gaps (PV max.)
Gravitational edge instabilities

GI associated with gaps or edges even when Toomre stability criterion satisfied ($Q_T > 1$ everywhere)

- Lovelace & Hohlfeld (1978); Sellwood & Kahn (1991): galactic/stellar disks
- Meschiari & Laughlin (2008): gaps in gaseous protoplanetary disks
- Lin & Papaloizou (2011b): confirmation of GEI for planet gaps (PV max.)

A necessary condition is

$$\Lambda = \beta \times \left| \frac{d^2}{dr^2} \left( \frac{\Omega \Sigma}{\kappa^2} \right) \right|_{\text{edge}} \sim Q^{-1} > 1$$

→ Don’t need small $Q_{\text{edge}}$. See Lin & Papaloizou (2011b) for details.
Gravitational edge instabilities

GI associated with gaps or edges even when Toomre stability criterion satisfied ($Q_T > 1$ everywhere)

- Lovelace & Hohlfeld (1978); Sellwood & Kahn (1991): galactic/stellar disks
- Meschiari & Laughlin (2008): gaps in gaseous protoplanetary disks
- Lin & Papaloizou (2011b): confirmation of GEI for planet gaps (PV max.)
Influence of GEI on disk-planet torques

Spirals supply material to execute horseshoe turns ahead of planet

Normal clean gap

Unstable gap edge

→ positive co-orbital torques
Influence of GEI on disk-planet torques

Spirals supply material to execute horseshoe turns ahead of planet

\[ Q_o = 1.7 \]

\[ [50, 80]P_0 \]

\[ [80, 110]P_0 \]

\[ (r - r_p)/r_h \]

(Lin & Papaloizou, 2012)
Outward migration induced by an unstable gap

$Q_o = 1.5$ and $Q_o = 1.7$ have GEI, $Q_o = 2.0$ does not (Lin & Papaloizou, 2012)
Dependency on planet mass

Instability ↔ gap structure ↔ planet mass ↔ orbital migration

[2012 CITA summer student project (Cloutier and Lin, 2013, submitted)]
Dependency on planet mass
Instability ↔ gap structure ↔ planet mass ↔ orbital migration

[2012 CITA summer student project (Cloutier and Lin, 2013, submitted)]
Torque balance?

Can positive torques counter-act inward type II migration → no migration?

Cloutier and Lin (2013, submitted)
Torque balance?

Can positive torque counteract inward type II migration → no migration?

Cloutier and Lin (2013, submitted)
Type III migration triggered by the unstable gap

Cloutier and Lin (2013, submitted)
Wide-orbit giant planet formation by disk fragmentation

E.g. HR 8799bcd, Fomalhault b (?)

- Zhu et al. (2012); Vorobyov (2013): most clumps fall in, but occasionally can survive by opening gaps
- Our simulations \(\rightarrow\) gap stability may be another issue
- Zhu et al.: additional clump formation along edge of a gap opened by a previous clump; Vorobyov: clump migrates outward

On the other hand:
- Move planets to large distances by inducing outward type III migration?
Future

- Gap formation/stability in non-isothermal disks (Lin and Cloutier, in preparation)
Future

- Gap formation/stability in non-isothermal disks (Lin and Cloutier, in preparation)
- Dead zone boundary GI (global transport)
Future

- Gap formation/stability in non-isothermal disks (Lin and Cloutier, in preparation)
- Dead zone boundary GI (global transport)

- Magneto-gravitational instabilities
References

Armitage P. J., 2011, ARAA, 49, 195